Zeldovich Flow on Cosmic Vacuum Background: New Exact Nonlinear Analytical Solution

Arthur D. Chernin[1,2,3], Dmitriy I. Nagirner[4], Svetlana V. Starikova[4]
[1]Sternberg Astronomical Institute, Moscow University, Moscow, 119899, Russia,
[2]Tuorla Observatory, University of Turku, Piikkiö, FIN-21500, Finland,
[3]Astronomy Division, Oulu University, Oulu, FIN-90014, Finland,
[4]Sobolev Astronomical Institute, St.-Petersburg State University,
Staryi Peterhoff, 198504, St.-Petersburg, Russia

Abstract

A new exact nonlinear Newtonian solution for a plane matter flow superimposed on the isotropic Hubble expansion is reported. The dynamical effect of cosmic vacuum is taken into account. The solution describes the evolution of nonlinear perturbations via gravitational instability of matter and the termination of the perturbation growth by anti-gravity of vacuum at the epoch of transition from matter domination to vacuum domination. On this basis, an 'approximate' 3D solution is suggested as an analog of the Zeldovich ansatz.

1 Introduction

Three decades ago, Zeldovich (1970) published in this Journal his now famous nonlinear theory of gravitational instability formulated in terms of Newtonian mechanics. The theory was first used for pancake cosmogony by Zeldovich and his group (see for a review Shandarin& Zeldovich 1989). Later on it was realized that the theory can be judiciously applied to a wide range of cosmological scenaria; in particular, the dynamics given by the theory underlies gravitational clustering (Peebles 1993). The Zeldovich theory provides also an effective analytical tool for optimization of cosmological simulations (Melott 1993).

The Zeldovich theory is based on an exact analytical solution which describes a plane pressure-free matter flow superimposed on the regular Hubble expansion:

$$x_1 = a(t)\chi_1 + b(t)\beta(\chi_1), \quad x_2 = a(t)\chi_2, \quad x_3 = a(t)\chi_3, \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$ are the Eulerian coordinates (Cartesian) of an element ('particle') with a Lagrangian coordinates $\boldsymbol{\chi} = (\chi_1, \chi_2, \chi_3)$, and $\beta(\chi_1)$ is an arbitrary function of only one coordinate χ_1. The first term in the first equation represents — together with the two other equations — the unperturbed isotropic solution for the Hubble flow, and the second term in the first equation represents a perturbation flow which depends on χ_1 and time t.

The solution was obtained by Zeldovich for the parabolic expansion (a spatially flat model), in which the scale factor $a(t) = a_0\left(\frac{3}{2}H_0 t\right)^{2/3}$, where a_0 and H_0 are the present values of $a(t)$ and Hubble constant, $b(t) = t^{4/3}$. The perturbation corresponds to the growing mode of Lifshits (1946) linear solution for gravitational instability of non-relativistic matter and has the same time dependence as in the linear theory.

A decreasing mode of Lifshits theory can also be introduced to the solution (Zentsova and Chernin 1980), and then one has instead of the first equation in Eq.(1):

$$x_1(\chi_1, t) = a(t)\chi_1 + b(t)\beta(\chi_1) + c(t)\gamma(\chi_1), \tag{2}$$

where $c(t) = t^{-1/3}$, and $\gamma(\chi_1)$ is the second arbitrary function of χ_1. For this more general solution, which is also nonlinear and exact, the density is

$$\rho = \frac{a_0}{6\pi G t^2}\left[a_0 + \left(\frac{3}{2}H_0\right)^{-2/3}\left(t^{2/3}\frac{\partial\beta}{\partial\chi_1} + t^{-1}\frac{\partial\gamma}{\partial\chi_1}\right)\right]^{-1}. \tag{3}$$

In Zeldovich (1970) work, a 3D 'approximation' is suggested, in which 3D perturbation flow is assumed to have the same structure and time dependence as the plane perturbation flow in the exact solution of Eq.(1). This ansatz has been justified by many numerical simulations (see again Shandarin and Zeldovich 1989). The corresponding exact analytical 3D nonlinear solution of the problem has not been found yet and there is little hope to find it.

In this Letter, we report a generalization of Zeldovich theory by accounting for the dynamical effect of the cosmological constant Λ, or cosmic vacuum. We give a new exact nonlinear Newtonian solution with the same symmetry as in Eqs.(1–3), but for a non-zero cosmological constant, or vacuum density $\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$. The observational basis for

this approach is due to the recent studies of distant type Ia supernovae (Perlmutter et al. 1998, Riess et al. 1999) which indicate that the vacuum density (in the units of the critical density) is $\Omega_\Lambda = 0.7 \pm 0.1$, which is in concordance with the bulk of observational data that come also from the cosmic age, the cosmic microwave background anisotropy in combination with cluster dynamics, etc. (Carrol et al. 2000, Wang et al. 2000).

In spherical symmetry, the nonlinear solution of the same problem (without vacuum) is given in the frame of the well-known Tolman-Bondi-Lemaître model (see Lahav et al. 1991 and the references therein).

2 Basic Equations

Following Zeldovich (1970), we consider a plane perturbation flow of pressure-free matter and search for a solution in the Lagrangian form

$$x_1 = a(t)\chi_1 + \delta(\chi_1, t), \quad x_2 = a(t)\chi_2, \quad x_3 = a(t)\chi_3. \tag{4}$$

The equation of motion for the 'unperturbed' function $a(t)$ in the background Friedman solution contains the matter and vacuum densities

$$\ddot{a} = -\frac{4\pi G}{3}\rho_{\mathrm{G}} a, \tag{5}$$

where the effective gravitating density

$$\rho_{\mathrm{G}} = \rho_0 \frac{a_0^3}{a^3} + \rho_\Lambda + 3\frac{P_\Lambda}{c^2} = \rho_0 \frac{a_0^3}{a^3} - 2\rho_\Lambda. \tag{6}$$

Here ρ_0 is the present-day mean matter density, P_Λ is the vacuum pressure, and the equation of state of vacuum is $P_\Lambda = -\rho_\Lambda c^2$.

The first integral of Eq.(5),

$$\dot{a}^2 = \frac{8\pi G}{3}\left(\rho_0 \frac{a_0^3}{a^3} + \rho_\Lambda\right) a^2 - kc^2, \tag{7}$$

is the Friedman cosmology equation, where $k = 1, 0, -1$ for elliptic,parabolic and hyperbolic dynamics, respectively.

The perturbation plane flow is considered in the Newtonian approximation. The basic equations for the flow of Eq.(4) are the Euler equation of motion, the continuity equation and the Poisson equation:

$$\dot{\boldsymbol{v}} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\nabla\varphi, \tag{8}$$
$$\dot{\rho} + \nabla(\rho\boldsymbol{v}) = 0, \tag{9}$$
$$\triangle\varphi = 4\pi G(\rho - 2\rho_\Lambda). \tag{10}$$

Here $\boldsymbol{v}$ and ρ are perturbed velocity and density. The dynamical effect of vacuum on the flow is taken into account in the Poisson equation by the vacuum density ρ_Λ which enters also the Eq.(5). The velocity component v_i depends on x_i only.

Now we use the Lagrangian coordinates instead of the Eulerian coordinates in accordance with the relations (prime denotes derivative on χ_1)

$$\frac{\partial}{\partial x_1} = \frac{1}{a+\delta'}\frac{\partial}{\partial \chi_1}, \tag{11}$$

$$\frac{\partial}{\partial x_2} = \frac{1}{a}\frac{\partial}{\partial \chi_2}, \quad \frac{\partial}{\partial x_3} = \frac{1}{a}\frac{\partial}{\partial \chi_3}, \tag{12}$$

$$\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x}} = \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{\chi}} - \frac{\dot{a}\chi_1 + \dot{\delta}}{a+\delta'}\frac{\partial}{\partial \chi_1} - \frac{\dot{a}}{a}\left(\chi_2\frac{\partial}{\partial \chi_2} + \chi_3\frac{\partial}{\partial \chi_3}\right). \tag{13}$$

As a result the continuity equation can be presented in the form

$$\frac{\dot{\rho}}{\rho} + \frac{\dot{a}+\dot{\delta}'}{a+\delta'} + 2\frac{\dot{a}}{a} = 0. \tag{14}$$

Its solution is

$$\rho = \frac{a_0^3}{a^2}\frac{\rho_0}{a+\delta'}. \tag{15}$$

Similarly Eq.(8) is reduced to the form

$$\nabla\varphi = -\dot{\boldsymbol{v}}. \tag{16}$$

Substituting this relation into the Poisson equation gives

$$-\frac{\ddot{a}}{a+\delta'} - 2\frac{\ddot{a}}{a} = 4\pi G(\rho - 2\rho_\Lambda). \tag{17}$$

With the use of Eqs.(5),(7) and (15), we obtain from Eq.(17) rather simple relation

$$\ddot{\delta}' = \frac{8\pi G}{3}\left(\rho_0\frac{a_0^3}{a^3} + \rho_\Lambda\right)\delta'. \tag{18}$$

Assuming, as usually, that this relation holds for both δ' and δ, we cross then from the derivatives with respect to t to the derivatives with respect to $s = a(t)/a_0$ and finally obtain the equation that controls the time behaviour of the flow:

$$\begin{aligned} &s^2\left(\rho_0 + \rho_\Lambda s^3 - \frac{kc^2}{a_0^2}\frac{3}{8\pi G}s\right)\frac{\mathrm{d}^2\delta}{\mathrm{d}s^2} \\ &- s\left(\frac{1}{2}\rho_0 - 2\rho_\Lambda s^3\right)\frac{\mathrm{d}\delta}{\mathrm{d}s} - (\rho_0 + \rho_\Lambda s^3)\delta = 0. \end{aligned} \tag{19}$$

Eq.(19) can be solved numerically with the use of a standard procedure. It is most important that for the parabolic ($k = 0$) motion, an exact solution of the equation can be found in an explicit analytical form.

3 Exact solution

Let us denote $q = \rho_\Lambda/\rho_0 = \Omega_\Lambda/\Omega_0$ where Ω_Λ and Ω_0 are the present vacuum and matter densities in the units of critical density. For $k = 0$ one has $\Omega_\Lambda + \Omega_0 = 1$ and

$$\Omega_\Lambda = \frac{q}{1+q}, \;\; \Omega_0 = \frac{1}{1+q}. \tag{20}$$

Introducing $u = qs^3 = q[a(t)/a_0]^3$, where now

$$a(t) = a_0 q^{-1/3} u^{1/3}(t), \;\; u(t) = \sinh^2\left(\frac{3}{2}\alpha t\right), \tag{21}$$

and $\alpha = \sqrt{\Omega_\Lambda} H_0 = \sqrt{q/(1+q)} H_0 = \sqrt{8\pi G \rho_\Lambda/3}$, we transform Eq.(19) to the form

$$9u^2(1+u)\frac{\mathrm{d}^2\delta}{\mathrm{d}u^2} + 9u\left(\frac{1}{2}+u\right)\frac{\mathrm{d}\delta}{\mathrm{d}u} - (1+u)\delta = 0, \tag{22}$$

Eq.(22) can be reduced to hypergeometric equation, and its general solution is given in terms of hypergeometric functions. Accordingly, one has the following solution:

$$\delta = B(qs^3) f_{\mathrm{B}}(\chi_1) + C(qs^3) f_{\mathrm{C}}(\chi_1). \tag{23}$$

The solution contains two modes of the perturbation flow (we call them 1 and 2) and includes f_{B} and f_{C} as arbitrary dimensionless functions of the Lagrangian coordinate χ_1. The second mode is given by an elementary function:

$$C(u) = \frac{\sqrt{1+u}}{u^{1/6}}. \tag{24}$$

As for the first mode, $B(u)$, it is given in terms of hypergeometric functions:

$$B(u) = \begin{cases} u^{2/3} F\left(1, \frac{1}{3}, \frac{11}{6}; -u\right) & \text{if} \quad 0 \le u < 1, \\ \frac{u^{2/3}}{1+u} F\left(1, \frac{3}{2}, \frac{11}{6}; \frac{u}{1+u}\right) & \text{if} \quad u \sim 1, \\ c_0 C(u) - \frac{5}{4} D(u) & \text{if} \quad u > 1, \end{cases} \tag{25}$$

where $c_0 = \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{11}{6}\right)\Gamma\left(\frac{2}{3}\right)$ and

$$D(u) = \frac{1}{u^{1/3}} F\left(1, \frac{1}{6}, \frac{5}{3}; -\frac{1}{u}\right) = \frac{4}{5}\left[c_0 C(u) - B(u)\right]. \tag{26}$$

Fig.1 demonstrates the behaviour of B, C and D as functions of time variable u (double logarithmic scale) in the most interesting "transition" range of the argument u, e.g. in the era, when the evolution of the flow transits from the initial epoch of dynamical domination of matter gravity to the later epoch of dynamical domination of vacuum anti-gravity.

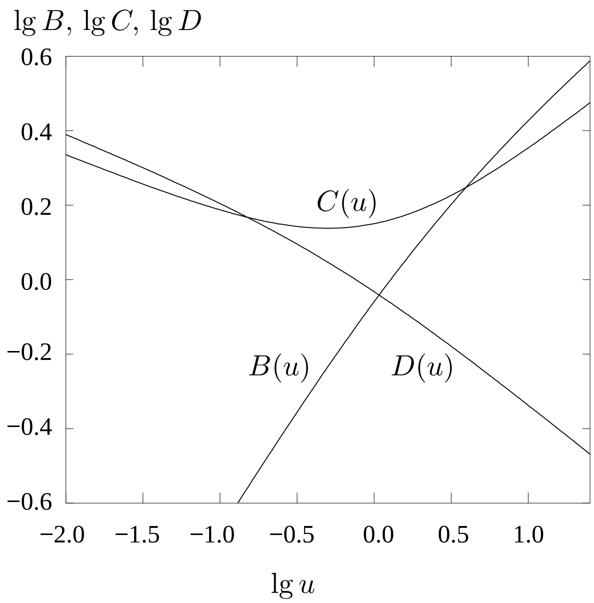

Fig.1. The functions $B(u)$, $C(u)$ and $D(u)$

It is also easy to see from Fig.1 the dependence of the functions on the redshift z, since $u = q(a/a_0)^3 = q(1+z)^{-3}$.

Simple asymptotic formulae for the three functions at the epochs before and after the transition era are given in Table 1. In this table the asymptotic dependence of a and u on time is also shown.

Table 1. Asymptotic behaviour of $a(t)$, $u(t)$, $B(u)$, $C(u)$, $D(u)$

	$t \to 0$	$t \to \infty$
$a(t)$	$a_0 q^{-1/3}(3\alpha t/2)^{2/3}$	$a_0 q^{-1/3} 2^{-2/3} e^{\alpha t}$
$u(t)$	$(3\alpha t/2)^2$	$2^{-2}\exp(3\alpha t)$
$B(u)$	$u^{2/3}$	$c_0 u^{1/3}$
$C(u)$	$u^{-1/6}$	$u^{1/3}$
$D(u)$	$4c_0 u^{-1/6}/5$	$u^{-1/3}$

4 Discussion

1. The exact solution (23) contains the Zeldovich solution of Eq.(1) and the exact solution of Eqs.(2,3) as particular cases for $\rho_\Lambda = 0$. It is also seen from Table 1 that Eqs.(1,2) represent an asymptotic of our solution in the limit $t \to 0$, $a(t) \to 0$, when the dynamical effect of vacuum can be neglected. Indeed, in this limit, the two modes of our solution vary as

$$B(u(t)) \propto b(t) = t^{4/3}, \quad C(u(t)) \propto c(t) = t^{-1/3}. \tag{27}$$

It may be easily found with this asymptotic that the arbitrary functions of the solution of Eqs.(1,2) are related to the arbitrary functions of our solution in a simple way:

$$\beta(\chi_1) = \left(\frac{3}{2}\alpha\right)^{4/3} f_{\rm B}(\chi_1), \quad \gamma(\chi_1) = \left(\frac{3}{2}\alpha\right)^{-1/3} f_{\rm C}(\chi_1). \tag{28}$$

2. In the opposite asymptotic region, in the limit when $t \to \infty$, $a(t) \to \infty$, vacuum dominates; since the gravity of matter can be then neglected, matter may be considered as a gas of test particles moving freely on the vacuum background. In this limiting case, the time behaviour of the perturbation flow differs drastically from the Zeldovich solution. Indeed, for large times one has from our solution:

$$\delta \propto B(u) \propto C(u) \propto a(t) \propto e^{\alpha t}. \tag{29}$$

As we see, the time behaviour of the perturbation flow is exactly the same as that of the unperturbed Hubble flow, in this limit. Accordingly, the whole nonlinear (and arbitrary non-uniform in density — see below) flow expands in a regular isotropic manner with the Hubble linear law.

3. Our solution allows a special case, in which functions f_B and f_C are related to each other as $f_\mathrm{B} = -f_\mathrm{C}/c_0 (= -4f_\mathrm{D}/5)$; the asymptotic of this special solution at $a(t) \to \infty$ is different from both Eqs.(1,2) and the equation above:

$$\delta \propto D(u) \propto a(t)^{-1} \propto e^{-\alpha t}. \tag{30}$$

The special solution describes the asymptotic "adiabatic cooling" of the nonlinear flow.

4. According to Eq.(15), the perturbation of the matter density relative to the perturbed density $\varepsilon = \dfrac{\delta\rho}{\rho} = -\dfrac{\delta'}{a}$. One may consider also the density perturbation relative to the unperturbed density:

$$\varepsilon_* = \frac{\delta\rho}{\rho_0}\frac{a_0^3}{a^3} = -\frac{\delta'}{a+\delta'} = \frac{a}{a+\delta'}\varepsilon. \tag{31}$$

The asymptotics of the relative density perturbation ε_*, when $t \to 0$ and $t \to \infty$, are given in Table 2 for the two modes of the general solution (denoted in the Table as 1,2) and for the special case 3.

Table 2. Asymptotics of ε_*

Mode	$t \to 0$	$t \to \infty$
1	$-f'_\mathrm{B}\dfrac{q^{1/3}}{a_0}\left(\dfrac{3}{2}\alpha t\right)^{2/3}$	$-f'_\mathrm{B}\dfrac{c_0 q^{1/3}}{a_0 + f'_\mathrm{B} c_0 q^{1/3}}$
2	-1	$-f'_\mathrm{C}\dfrac{q^{1/3}}{a_0 + f'_\mathrm{C} q^{1/3}}$
3	-1	$-f'_\mathrm{D} 2^{4/3}\dfrac{q^{1/3}}{a_0} e^{-2\alpha t}$

If one considers ε at small times, it may be seen that its behaviour is the same as ε_* for the first mode of the solution. For the the second mode $\varepsilon \sim -f'_\mathrm{C}\dfrac{q^{1/3}}{a_0}\left(\dfrac{3}{2}\alpha t\right)^{-1}$. For the third mode we must change f_C to f_D and add factor $4c_0/5$.

When $t \to \infty$ then ε coincides with ε_* for mode 3 and does not contain the second terms in denominators of modes 1, 2.

5. In the velocity perturbation, there is only one component, $\delta v_1 = \dot{\delta}$, and the relative velocity perturbation is

$$\delta_* v_1 = \frac{\delta v_1}{\chi_1 \dot{a}} = \frac{1}{\chi_1}\frac{\partial \delta}{\partial a}. \tag{32}$$

The asymptotics of relative perturbations of velocity for the same three cases 1,2,3, as in Table 2, are given in Table 3.

Table 3. Asymptotics of $\delta_* v_1$

Mode	$t \to 0$	$t \to \infty$
1	$\frac{f_{\rm B}}{\chi_1} q^{1/3}\left(\frac{3}{2}\alpha t\right)^{2/3}$	$\frac{f_{\rm B}}{\chi_1} c_0 q^{1/3}$
2	$-\frac{f_{\rm C}}{\chi_1}\frac{q^{1/3}}{3}(\alpha t)^{-1}$	$\frac{f_{\rm C}}{\chi_1} q^{1/3}$
3	$-\frac{4}{15}\frac{f_{\rm D}}{\chi_1} c_0 q^{1/3}(\alpha t)^{-1}$	$-\frac{f_{\rm D}}{\chi_1} 2^{4/3} q^{1/3} e^{-2\alpha t}$

(We omit the argument χ_1 of functions f in Table 3 and of their derivatives in Table 2.)

This analysis shows that gravitational instability is terminated and the nonlinear perturbations, that are growing at earlier epoch, are then freezing out or even being "adiabatically cooled", when vacuum starts to dominate dynamically; this effect is clearly due to anti-gravity of vacuum.

6. It is worth to find a relation between the nonlinear solution and the behaviour of linear perturbations on the vacuum background. The general solution for linear perturbations may easily be found with the method suggested by Zeldovich (1965) for the Lifshits-type perturbations (see for comparison Heath 1977, Lahav et al. 1991):

$$\delta = F_1(\chi_1)\dot{a}(t)\int \frac{dt}{\dot{a}^3(t)} + F_2(\chi_1)\dot{a}(t), \tag{33}$$

where F_1, F_2 are arbitrary functions of χ_1. Two these linearly independent linear solutions are directly related to modes 1,2 of the nonlinear solution obtained here if we use $a(t)$ from (21):

$$\dot{a}(t)\int_0^t \frac{\mathrm{d}t}{\dot{a}^2(t)} = \frac{2}{5}\frac{q^{1/3}}{a_0\alpha^2}B(u(t)), \dot{a}(t) = a_0 q^{-1/3}\alpha C(u(t)). \tag{34}$$

For Mode 3 one has:

$$D(u(t)) = 2\alpha^2 a_0 q^{-1/3}\dot{a}\int_t^\infty \frac{\mathrm{d}t}{\dot{a}^2(t)}. \tag{35}$$

7. Finally, on the basis of our solution, a 3D 'approximate' (in Zeldovich's (1970) sense) solution may be suggested as a generalization of the Zeldovich ansatz by taking into account the dynamical effect of cosmic vacuum:

$$\boldsymbol{x}(\boldsymbol{\chi}, t) = a(t)\boldsymbol{\chi} + B(t)\boldsymbol{f}_1(\boldsymbol{\chi}) + C(t)\boldsymbol{f}_2(\boldsymbol{\chi}), \tag{36}$$

where the time functions B and C are given by our nonlinear solution, $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ are arbitrary vector functions of the Lagrangian 3D coordinates $\boldsymbol{\chi}$.

We expect that this new generalized 3D approximation can effectively be used for the theories of the large-scale structure formation in ΛCDM cosmologies, – in the same way as the Zeldovich original ansatz has been used for cosmologies with zero cosmological constant.

It will be interesting to investigate the influence of radiation on the evolution of perturbations.

Acknowledgements This work was supported in part by the Russian Leading Scientific Schools grant 00-15-96607 and Federal Program "Integration" project N 578. A.C. appreciates a partial support from the Academy of Finland grant "Galactic streams and dark matter structures".

References

[2000] Carrol S.M., 2000, astro-ph/0004075.

[1977] Heath D.J., 1977, MN RAS 179, 351.

[1991] Lahav O., Lilje P.B., Primack J.R., Rees M.J., 1991, MN RAS 251, 128.

[1946] Lifshits E.M., 1946, JETP 16, 587.

[1993] Melott A.L., 1993, Comm. Astrophys. 17, 145.

[1993] Peebles P.J.E., 1993, Principles of Physical Cosmology, Princeton University Press.

[1999] Perlmutter S. et al., 1999. ApJ, 517, 565.

[1998] Riess A., Filipenko A.V., Challis P., et al., 1998, AJ 116, 1009.

[1989] Shandarin, S.F., Zel'dovich, Ya.B., 1989, Rev. Mod. Phys. 61, 185.

[2000] Wang L. et al., 2000, Astrophys. J. 530, 17.

[1977] Zentsova A.C., and Chernin A.D., 1980, Astrophysics 16, 108.

[1965] Zeldovich Ya.B., 1965, Adv. Astron. Ap. 3, 241.

[1970] Zeldovich Ya.B., 1970, A&A 5, 84.